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On the Solvability of Certain Factorizable Groups

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1. INTRODUCTION

Let G be a group and $G = AB$, where A and B are subgroups of G. There are a number of results which deduce the solvability of G from suitable conditions on A and B. In particular, in several of these results the conditions are that A and B each contain a subgroup of index 2 of particular form: for example, if G is finite and A and B are both dihedral or dicyclic, then G is solvable [2, 6]. Also see recent work of Knop [5]. We prove a result of this type, namely

MAIN THEOREM. *Let G be a finite group with subgroups A, B, H, and K such that $G = AB$, $H \leqslant A$, $K \leqslant B$, $[A : H] = [B : K] = 2$ and H is nilpotent and K is a p-group for some prime p. Then G is solvable.*

2. PRELIMINARIES

All groups considered are finite. The notations are standard. In particular, $X \leqslant Y$ means X is a subgroup of Y with $X < Y$ and $X \trianglelefteq Y$ meaning proper and normal subgroups, respectively, and if X is any subset of G, then $\langle X \rangle$ means the subgroup of G generated by the elements of X. Also $\pi(G)$ is the set of primes dividing the order of G. Many of the results we quote can be found in [7, especially Chap. 13].

We use throughout the Kegel–Wielandt theorem [4, 9] that the product of two finite nilpotent groups is solvable.

In addition, we need a lemma, which is essentially the proof of Kegel's result. We follow the proof given in [7, p. 381].

LEMMA. *Let G be a finite group, A and B subgroups of G with $G = AB$ and suppose every proper quotient group of G, and every proper subgroup of G containing A or B is solvable. If there is a prime p such that $p \mid |A|$ and $p \mid |B|$*

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and if there are Sylow p-subgroups P and Q of A and B, normal in A and B, respectively, then G is solvable.

Proof. Let $N = P^G$, the normal closure of P in G. If $AN < G$, then AN is solvable, since it is a proper subgroup containing A, so N is solvable as well. Since G/N is solvable by the hypotheses of the lemma, this makes G solvable. Thus we may assume $AN = G$ and similarly $BN = G$, and $G/N = AN/N = BN/N$ has Sylow p-subgroups PN/N and QN/N. Since $P \leqslant N$, however, we have $PN = N$ and so $QN = N$. Thus $Q \leqslant N$, so $Q^G \leqslant N = P^G$. By symmetry we also have $P^G \leqslant Q^G$, so $P^G = Q^G$.

Since P and Q are the unique Sylow p-subgroups of A and B, respectively, it follows [8, Satz 6] that PQ is a subgroup of G, and from the normality of P and Q in A and B, respectively, it follows by [9, Lemma 10] that PQ^t is a subgroup for all t in G. Let P_1 be a p-subgroup of N containing P maximal with respect to the property that P_1Q^t is a subgroup of G for all $t \in G$. Then $P_1{}^G = N$.

If P_1 is normal in N then N is generated by p-subgroups normal in N, making N itself a p-group and therefore solvable. Since G/N is solvable by hypothesis, G is solvable in this case.

Thus we may assume that P_1 is not normal in N. Since $Q^G = N$, there exist $x, y \in G$ such that $y \in Q^x$ and $y \notin N(P_1)$. Since P_1Q^t is a subgroup for all $t \in G$, also $P_1{}^yQ^t$ is a subgroup, since $P_1{}^yQ^t = (P_1Q^{ty^{-1}})^y$. Therefore $\langle P_1, P_1{}^y\rangle Q^t$ is a subgroup for all $t \in G$. But from the choice of y, $P_1 < \langle P_1, P_1{}^y\rangle \leqslant N$, so from the maximality of P_1, we conclude $\langle P_1, P_1{}^y\rangle$ is not a p-group.

However, $\langle P_1, P_1{}^y\rangle \leqslant \langle P_1, P_1{}^y\rangle Q^x = \langle P_1, Q^x\rangle = P_1Q^x$, which is a p-group. This contradiction completes the proof of the lemma.

We now prove the first of two preliminary results.

THEOREM 1. *If G is a finite group, A, B, and H subgroups of G such that $G = AB$, $H \leqslant A$ and $[A : H] = 2$, H is nilpotent and B is a p-group for some prime p, then G is solvable.*

Proof. Suppose G is a counterexample of minimal order. If N is any nontrivial normal subgroup of G, $G/N = AN/N \cdot BN/N$, and BN/N is a p-group and AN/N contains a subgroup HN/N of index 1 or 2, so either AN/N is nilpotent or has a nilpotent subgroup of index 2. Then G/N is solvable by the Kegel–Wielandt theorem or by induction. If M is any proper subgroup of G with $A \leqslant M$, then by the Dedekind identity $M = A \cdot (M \cap B)$ and so M is solvable by induction. Similarly, any proper subgroup of G containing B is solvable by the Kegel–Wielandt theorem or induction.

For any prime q dividing $|H|$, let H_q be the Sylow q-subgroup of H. Since H_q is a characteristic subgroup of H, which is normal in A, we have $H_q \lhd A$. If $q \neq 2$, H_q is actually a normal Sylow q-subgroup of A. Thus A has a normal 2-complement, $H_{2'}$.

We consider several cases, depending on the prime p.

Case 1. $p = 2$. Consider $G_2 \cdot H_{2'}$, where G_2 is a Sylow 2-subgroup of G. Since each prime of $|G|$ divides $|G_2 \cdot H_{2'}|$ to its full power in $|G|$, we must have $G = G_2 \cdot H_{2'}$, and G is solvable by the Kegel–Wielandt theorem.

Case 2. $p \mid |A|$ and $p \neq 2$. All the conditions of the lemma are satisfied, and so G is solvable in this case also.

Case 3. $p \nmid |A|$ and $p \neq 2$. (i) G must be non-Abelian simple. For N any nontrivial normal subgroup of G, consider AN which is a subgroup of G. If $AN < G$, then AN is solvable as $A \leqslant AN$, and, therefore, N is solvable as $N \leqslant AN$. Since G/N is also solvable, this would imply that G is solvable. So $AN = G$. Similarly $BN = G$. Since in this case $|A|$ and $|B|$ are relatively prime, we have $|G| = |A| \cdot |B|$, and so $|B| \mid |N|$ and $|A| \mid |N|$, which imply $|G| \mid |N|$ and $G = N$. Thus G is simple.

(ii) We must have $2 \mid |H|$. If not, the Sylow 2-subgroups of G have order 2, and so G cannot be non-Abelian simple, contrary to (i).

(iii) Let A_2 be a Sylow 2-subgroup of A. Since $H_2 \lhd A_2$, we have $H_2 \cap Z(A_2) > 1$. But if $1 \neq x \in H_2 \cap Z(A_2)$, then x commutes with $H_{2'}$, since H is nilpotent, and with A_2, so it commutes with $A = A_2 \cdot H_{2'}$. Thus $x \in Z(A)$. Then the conjugacy class of x in G has p-power size, since $|A| \mid |C_G(x)|$, and so by a theorem of Burnside, G cannot be simple, contrary to (i). This completes the proof of the theorem.

We now reverse the roles of H and B.

THEOREM 2. *If G is a finite group, A, B, and H subgroups of G such that $G = AB$, $H \leqslant A$ and $[A : H] = 2$, H is a p-group for some prime p, and B is nilpotent, then G is solvable.*

Proof. Let G be a counterexample of minimal order. As in Theorem 1, every proper quotient group of G and every proper subgroup of G containing A or B is solvable. We now consider:

Case 1. $p = 2$. If $p = 2$, then A is a 2-group and so G is solvable by the Kegel–Wielandt theorem.

Case 2. $p \mid |B|$. By the lemma, $p \mid |B|$ implies that G is solvable.

Case 3. $p \nmid |B|$ and $p \neq 2$. We must have $2 \mid |B|$. If $2 \nmid |B|$, then $|A|$ and $|B|$ are relatively prime. As in Theorem 1, this implies that G is non-Abelian simple. But since in this case a Sylow 2-subgroup of G has order 2, G is not non-Abelian simple, and so $2 \mid |B|$.

Now, let $K = N_G(B_2)$, where B_2 is the Sylow 2-subgroup of B. If $K = G$, then B_2 is a solvable normal subgroup of G, making G solvable. Thus $K \neq G$.

Now $B \leqslant K$ since B is nilpotent, and so K is solvable. Let C be a Hall $\pi(B)$-subgroup of K containing B. Now if $A \cap B > 1$, we would have $|A \cap B| = 2$ and then $G = HB$ and G would be solvable by the Kegel–Wielandt theorem. Thus $A \cap B = 1$. Therefore, if G_2 is a Sylow 2-subgroup of G containing B_2, we have $[G_2 : B_2] = 2$, and so K contains a full Sylow 2-subgroup of G and $2 | B_2 | \, \big| \, | C |$. Also $| B | \, \big| \, | C |$, and since $p \nmid | B |$, we have $p \nmid | C |$. Therefore $| C | = 2 | B |$. By calculating orders, we have $G = CH$. Now since H is a p-group and C has a nilpotent subgroup of index 2, G is solvable by Theorem 1, and the proof of the theorem is complete.

The simple group of order 168, PSL(2, 7), is the product of a group of order 21 and a group of order 8. Thus in Theorems 1 and 2 the condition $[A : H] = 2$ cannot be replaced by $[A : H]$ is any prime. However, the proofs are valid, essentially unchanged, if this condition is replaced by $[A : H] = q$, where q is the smallest prime dividing the order of G. (Of course, then G would be solvable for having odd order.) In the proof of the main theorem, however, the prime 2 plays a critical role.

3. Proof of Main Theorem

Let G be a counterexample of minimal order. By the Kegel–Wielandt theorem, or by Theorem 1, Theorem 2, or by induction (briefly "by induction"), every proper quotient group of G and every proper subgroup of G containing A or B is solvable. Also G has no solvable normal subgroups. We show

(1) $p \neq 2$. If $p = 2$, then B is a 2-group and so G is solvable by Theorem 1.

(2) $p \nmid | H |$. If $p \, \big| \, | H |$, let P be a Sylow p-subgroup of H. Since H is nilpotent, P is a characteristic subgroup of H, which is normal in A, so P is a normal p-subgroup of A. Also, K is a normal Sylow p-subgroup of B. The group G now satisfies all the conditions of Kegel's lemma, and so G is solvable, contradicting the choice of G.

(3) $(| A |, | B |) = 2$. This follows from (2) and the fact that $| A | = 2 | H |$ and $| B | = 2 | K |$.

(4) $A \cap B = 1$ and $| G | = | A | \cdot | B |$. If $A \cap B > 1$, then by (3), $A \cap B$ has order 2, and is a Sylow 2-subgroup of B, say $A \cap B = B_2$. Then $B = B_2 \cdot K$, and so $G = AB = AB_2K = AK$ and is solvable by Theorem 1. Therefore $| A \cap B | = 1$ and $| G | = | A | \cdot | B | / | A \cap B | = | A | \cdot | B |$.

(5) There exists a normal subgroup N of G, with $[G : N] = 2$.

Let $\mathscr{S}$ be the set of right cosets of A in G, and consider the action of G on $\mathscr{S}$. If this action is not faithful, the kernel is a nontrivial normal subgroup of G which is contained in A and therefore solvable, making G solvable. Hence the action is faithful. Now since $A \cap B = 1$, B acts regularly on $\mathscr{S}$, and $\mathscr{S}$ has $[G : A] = |B|$ elements which is twice an odd number. Thus an involution in B is represented as an odd number of disjoint transpositions, and hence as an odd permutation. So G contains an element represented as an odd permutation, and so G has a normal subgroup of index 2, which we call N (see [6]).

Of course, since G is not solvable and any proper quotient group of G is solvable, N is not solvable.

(6) N is the only proper normal subgroup of G. Let M be any proper normal subgroup of G. Then AM is a subgroup of G, and contains A, so if $AM < G$, AM is solvable by induction, and so M is solvable, making G solvable. Hence $AM = G$. Similarly $BM = G$. From $AM = G$ and (4) follows $|B| \bigm| |M|$, and similarly $|A| \bigm| |M|$. Since $(|A|, |B|) = 2$, this implies $\frac{1}{2}|G| \bigm| |M|$, so $|M| = \frac{1}{2}|G|$ or $|G|$.

Now suppose $|M| = \frac{1}{2}|G|$. We wish to show $M = N$ (that is, that any two normal subgroups of this order are equal). Suppose $M \neq N$. Then since $M \cap N$ is a normal subgroup of G, $M \cap N = 1$ by the paragraph above. Then $|M| \cdot |N| = |MN| = |G| = 2|M|$, and so $|N| = 2$, and $|G| = 4$ and so G is solvable, a contradiction. Hence (6) follows.

(7) N is the direct product of isomorphic non-Abelian simple groups. Since N is the only proper normal subgroup of G, N has no proper characteristic subgroups, and N is not solvable, and so N has the claimed structure [7, Theorem 4.4.3].

We now fix some notation. By [8], there exist Sylow 2-subgroups G_2, A_2, and B_2 of G, A, and B, respectively, such that $G_2 = A_2B_2$. Since N is normal in G, there exists a Sylow 2-subgroup of N, N_2 contained in G_2. Also, let $H_{2'}$ be the unique Hall $2'$-subgroup of H, and let H_2 be the Sylow 2-subgroup of H. Note that H_2 and $H_{2'}$ are normal in A.

(8) $H_{2'} \leqslant N$, $K \leqslant N$, $A_2 \nleqslant N$, $B_2 \nleqslant N$. The first two just follow from the fact that N is a normal subgroup of G of index 2. If A_2 or B_2 were contained in N, then A or B would be contained in N, and by induction N would be a solvable normal subgroup of G, a contradiction.

(9) $2 \bigm| |H|$, since otherwise $|G_2| = 4$, and $|N_2| = 2$, which implies that N is actually simple (and not a direct product); but a non-Abelian simple group cannot have a Sylow 2-subgroup of order 2. Recall that we have $[G_2 : A_2] = [A_2 : H_2] = 2$.

(10) The following cannot occur: N simple, N_2 dihedral of order 4 or 8, and $H_2N_2 = G_2$. Assume to the contrary that this does occur. Note this implies $HN = G$, and so if we call $|N| = 2^\alpha p^\beta n$, where $\alpha = 2$ or 3, and $(2, n) = (p, n) = 1$, then we can calculate $|H \cap N| = 2^{\alpha-2}n$ and so $H \cap N$ is a nilpotent subgroup of N of index $4p^\beta$.

Now by Gorenstein and Walter's classification of all simple groups with dihedral Sylow 2-subgroup [1], we must have N isomorphic to PSL(2, q^f), $q^f > 3$, q a prime, or to A_7. However, none of these is possible. In A_7, a subgroup satisfying the conditions on $H \cap N$ would be Abelian; A_7 has no such subgroup. In PSL(2, q^f), we can use Dickson's determination of the subgroups of PSL(2, q^f) [3, Theorem II.827], and see that none of these groups has a subgroup satisfying the conditions on $H \cap N$, except PSL(2, 4) and PSL(2, 5), with $|H \cap N| = 3$ or 5. Of course PSL(2, 4) $\cong$ PSL(2, 5) $\cong A_5$, the alternating group on five letters. But if $N \cong A_5$, then $G \cong S_5$, the symmetric group on five letters, as this is the only group of order 120 having a subgroup isomorphic to A_5 as its only normal subgroup. However, S_5 has no factorization of the type in the hypotheses in this theorem. This establishes (10).

(11) $H_2 \cap N_2 > 1$. If $H_2 \cap N_2 = 1$, then $2 \mid N_2 \mid = \mid G_2 \mid = \mid H_2N_2 \mid = \mid H_2 \mid \cdot \mid N_2 \mid$ and so $\mid H_2 \mid = 2$. Therefore, $\mid G_2 \mid = 8$ and $\mid N_2 \mid = 4$. Then N cannot be the direct product of more than one non-Abelian simple group, since each would have a Sylow 2-subgroup of order 2. Hence N must be simple. Then N_2 cannot be cyclic, as a non-Abelian simple group cannot have a cyclic Sylow 2-subgroup [7, Theorem 6.2.11]. Thus N_2 is dihedral of order 4 and $H_2N_2 = G_2$, contrary to (10). This establishes (11).

(12) There is a subgroup C, with $1 < C \leqslant H_2 \cap N_2$, and C normal in N_2.

If H_2 is normal in G_2, then $H_2 \cap N_2$ is normal in G_2, and since $H_2 \cap N_2 > 1$ by (11), we have $C = H_2 \cap N_2$ as required.

If H_2 is not normal in G_2, then $N_{G_2}(H_2) = A_2$, and since $[G_2 : A_2] = 2$, H_2 has exactly two conjugates in G_2, say H_2 and H_2^x, for any $x \in G_2 - A_2$. Also, since $A_2 \lhd G_2$, we have $H_2^x \leqslant A_2$, so $H_2H_2^x = A_2$ and

$$2 \mid H_2 \mid = \mid A_2 \mid = (\mid H_2 \mid \mid H_2^x \mid)/(\mid H_2 \cap H_2^x \mid),$$

which implies $\mid H_2 \mid = 2 \mid H_2 \cap H_2^x \mid$ and so $[H_2 : H_2 \cap H_2^x] = 2$.

Note that $H_2 \cap H_2^x$ is a normal subgroup of G_2, as it is the intersection of all the conjugates of H_2 in G_2.

We now consider several cases, depending on the order of $H_2 \cap H_2^x$:

(i) $\mid H_2 \cap H_2^x \mid = 1$. In this case $\mid H_2 \mid = 2$ so $\mid N_2 \mid = 4$. Then N cannot be the direct product of more than one non-Abelian simple group, since each would have Sylow 2-subgroup of order 2, so N must be simple. Since N_2 cannot be cyclic, as above, N_2 must be dihedral. Since $N_2 \lhd G_2$, we have H_2N_2 a subgroup of G_2, and either $H_2N_2 = N_2$ or $H_2N_2 = G_2$. The latter cannot occur by (10). If $H_2N_2 = N_2$, then $H_2 \cap N_2$ has order 2 and $H_2 \cap N_2 \lhd N_2$ since it has index 2; thus in this case we let $C = H_2 \cap N_2$.

(ii) $| H_2 \cap H_2^x | = 2$. In this case $| N_2 | = 8$ and, as in (i), N must be simple. Again $H_2N_2 = N_2$ or $H_2N_2 = G_2$. If $H_2N_2 = N_2$, then $H_2 = H_2 \cap N_2$, and $H_2 \cap H_2^x \leqslant H_2 \cap N_2$, and $C = H_2 \cap H_2^x$ is as required in this case.

If $H_2N_2 = G_2$, then $| H_2 : H_2 \cap N_2 | = 2$, so $| H_2 \cap N_2 | = 2$. Since N_2 has order 8, N_2 is either Abelian, quaternion, or dihedral. N_2 cannot be dihedral, by (10). If N_2 is either Abelian or quaternion, the subgroup $H_2 \cap N_2$ of order 2 is normal in N_2, and so $C = H_2 \cap N_2$ is as required in this case.

(iii) The remaining case is $| H_2 \cap H_2^x | \geqslant 4$. Since $H_2 \cap H_2^x \leqslant G_2$ and $| G_2 : N_2 | = 2$ we have $| H_2 \cap H_2^x \cap N_2 | \geqslant 2$, so $C = H_2 \cap H_2^x \cap N_2$ is as required (in fact, $C \lhd G_2$). Thus N_2 has a normal subgroup C, as claimed.

(13) Conclusion. Since $1 < C \lhd N_2$, we have $1 < C \cap Z(N_2)$. Let $x \in C \cap Z(N_2)$, $x \neq 1$. Then $N_2 \leqslant C_N(x)$; since $x \in H_2$ and H is nilpotent, $H_{2'} \leqslant C_N(x)$. Thus $[N : C_N(x)]$ is a power of p, since all the other primes of N divide $C_N(x)$ to their full power in N, and so the size of the conjugacy class of x in N is a power of p. Thus N cannot be non-Abelian simple by the theorem of Burnside quoted above.

Thus N must be a direct product of non-Abelian simple groups; let N^* be one of those non-Abelian simple groups for which the projection of x is not the identity. Call the projection $\pi: N \to N^*$. Then $C_{N^*}(\pi(x)) = \pi(C_N(x))$. Since $N_2 \leqslant C_N(x)$, we have that $\pi(N_2)$, which is a Sylow 2-subgroup of N^*, is contained in $C_{N^*}(\pi(x))$. Similarly $C_{N^*}(\pi(x))$ contains Sylow subgroups of N^* for all primes of N^* different from p. Therefore, $[N^* : C_{N^*}(\pi(x))]$ is a power of p, which by Burnside's theorem, contradicts N^* being non-Abelian simple.

This final contradiction completes the proof of the Main Theorem.

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